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ON THE APPLICATION OF THE METHOD OF LEAST SQU'S TO THE REDUCT'N OF COMPARISONS OF LINE-MEASURES AND TO THE CALIBRATION OF THERMOMETERS.

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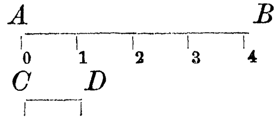
1. Comparisons of line-measures and observations for the calibration of thermometers may be discussed together, as there is no essential difference in the mode of making the observations or of reducing them.



Let *AB* be the line-measure divided into equal parts as nearly as may be at the points 1, 2, 3, ... The problem is to find the corrections to the graduation marks at the several points of division. In making the necessary observations, microscopes are mounted over the points 0 and 1 and readings made. The bar is then moved along until the points 1,2; 2,3,... are under the microscopes and readings are made each time. For the purpose of facilitating the work an extra scale *CD* may be employed and comparisons made between it and each of the spaces 0 1, 1 2, ... in succession. The correct's then are the amounts by which the marks should be changed in order to be in their true positions. When the corrections have been applied to the distances 0 1, 1 2,... these distances are all reduced to the same length.

With a thermometer *AB*, if 0, 1, 2, ... denote graduation marks, then by breaking off a column of mercury approximately equal in length to the distance 0 1, we may by reading the ends of this column when in the positions 0 1, 1 2,... determine the degree of uniformity of the bore of the tube, whence may be derived at the several points the *calibration corrections*, as they are called. The corrections to the graduation marks for calibration when applied to the observed values of the spaces 0 1, 1 2,... change them to what would have been found had the tube been of uniform bore throughout its whole length.

2. As the precision with which comparisons of line-measures can now be made is very great, it is fitting that the reduct'n of such comparisons should be carried out by the method of least-squares. With thermometers however such niceness of reading has not yet been attained and approximate methods of reduction may consequently be used to advantage. By such methods results will be found agreeing as closely with those found by the rigorous method as the quality of the observations will warrant. In investigating the reliance to be placed on approximate methods we must understand the rigorous method to see just what liberties we do take.

3. In a line-measure AB, let the spaces 0 1, 1 2, . . . be compared with the auxiliary space CD and let x_0, x_1, . . . be the graduation corrections at the points 0, 1, . . . Then, for the first comparison, if R_0, R_1 be the microscope readings at C, D and M_0, M_1 be the readings at 0, 1 we have

$$(M_1+x_1)-(M_0+x_0)-[(R_1-R_0)+y]=v,$$

where y is the amount by which CD differs from the space intended to be represented by CD and v is the residual error of observation.

Hence, for simplicity taking 4 spaces only, the observation equations may be written

$$\left.\begin{array}{r} x_1-x_0-y-(0\,1)=v_1 \\ x_2-x_1\quad -y-(1\,2)=v_2 \\ x_3-x_2\quad -y-(2\,3)=v_3 \\ x_4-x_3\quad -y-(3\,4)=v_4 \end{array}\right\}, \tag{1}$$

where $(0\,1)=-(M_1-M_0)+(R_1-R_0)$

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As these equations contain 6 unknowns and are themselves only four in number, the number of unknowns must be arbitrarily reduced in order to carry out the solution by the method of least squares. The simplest supposition is to make $x_0=0$, $x_4=0$, $y=$ a known quantity. All of the readings being supposed to have been equally well made and the observation equations to be of the same weight, we have the normal equations

$$\left.\begin{array}{rcl} 2x_1-x_2 & = & (0\,1)'-(1\,2)' \\ -x_1+2x_2-x_3 & = & (1\,2)'-(2\,3)' \\ -x_2+2x_3 & = & (2\,3)'-(3\,4)' \end{array}\right\}, \tag{2}$$

from which the unknowns are found.

Writing $[al]$, $[bl]$, $[cl]$ for the constant terms of the normal equations and solving

$$\left.\begin{array}{l} x_1=\tfrac{1}{4}(3[al]+2[bl]+[cl]) \\ x_2=\tfrac{1}{4}(2[al]+4[bl]+2[cl]) \\ x_3=\tfrac{1}{4}([al]+2[bl]+3[cl]) \end{array}\right\} \tag{3}$$

Hence weight of $x_1=\frac{4}{3}$, of $x_2=\frac{4}{4}$, of $x_3=\frac{4}{3}$.

4. Generally if the number of spaces is n,

$$\begin{array}{rcl} n.x_1 & = & (n-1)[al]+\ (n-2)[bl]+\ (n-3)[cl]+\ -\ -\ - \\ n.x_2 & = & (n-2)[al]+2(n-2)[bl]+2(n-3)[cl]+\ -\ -\ - \\ n.x_3 & = & (n-3)[al]+2(n-2)[bl]+3(n-3)[cl]+\ -\ -\ - \\ - & - & -\ \ -\ \ -\ \ -\ \ -\ \ -\ \ -\ \ -\ \ -\ -\ - \end{array}$$

Hence the weight of any correction as x_m is

$$\frac{n}{m(n-m)}.$$

Since this expression remains unchanged when $n-m$ is written for m, and is a minimum when $m=\frac{1}{2}n$, it follows that for graduation errors equally distant from the ends the weights are equal, and the weight decreases as we approach the center where it is a minimum.

5. It is in accordance with general experience and the principles of least squares that instead of spending all of the time of observation on the direct measurement of the spaces 0 1, 1 2, . . . better results would be obtained by spending part of it in reading combinations of these spaces as 0 2, 2 4, . . .; 0 3, 1 4, . . . Let then, for uniformity, each space and all possible combinations of spaces be equally well read. This will require instead of a single auxiliary space *CD* an auxiliary bar divided as nearly as possible the same as the bar *AB*. The observation equations therefore when written in full are

$$\left.\begin{array}{rrrrrrrl} & & & x_1 & -x_0 & -y_1 & & -(0\,1) = v_1 \\ & & x_2 & -x_1 & & -y_1 & & -(1\,2) = v_2 \\ & x_3 & -x_2 & & & -y_1 & & -(2\,3) = v_3 \\ x_4 & -x_3 & & & & -y_1 & & -(3\,4) = v_4 \\ & & x_2 & & -x_0 & -y_2 & & -(0\,2) = v_5 \\ & x_3 & & -x_1 & & -y_2 & & -(1\,3) = v_6 \\ x_4 & & -x_2 & & & -y_2 & & -(2\,4) = v_7 \\ & x_3 & & & -x_0 & & -y_3 & -(0\,3) = v_8 \\ x_4 & & & -x_1 & & & -y_3 & -(1\,4) = v_9 \\ x_4 & & & & -x_0 & & -y_4 & -(0\,4) = v_{10} \end{array}\right\} \quad (4)$$

10 equations with 9 unknowns.

6. Two cases may arise. In the first place the values of the spaces on the auxiliary bar *CD* may be known in terms of the accepted standard. We had an example of this on the Lake Survey when the $\frac{1}{2}$ mm. spaces on the Repsold Meter were compared with spaces on the Troughton & Simms Inch. The values of the spaces on the inch had been previously determined.

In this case then y_1, y_2, y_3, y_4 are known and the observation equations may be written more simply

$$\left.\begin{array}{r}
x_1-x_0-(0\,1) = v_1 \\
x_2-x_1\quad-(1\,2) = v_2 \\
x_3-x_2\quad-(2\,3) = v_3 \\
x_4-x_3\quad-(3\,4) = v_4 \\
x_2\quad-x_0-(0\,2) = v_5 \\
x_3\quad-x_1\quad-(1\,3) = v_6 \\
x_4\quad-x_2\quad-(2\,4) = v_7 \\
x_3\quad-x_0-(0\,3) = v_8 \\
x_4\quad-x_1\quad-(1\,4) = v_9 \\
x_4\quad-x_0-(0\,4) = v_{10}
\end{array}\right\}. \qquad (5)$$

The normal equations are

$$\left.\begin{array}{rcl}
4x_0-x_1-x_2-x_3-x_4 &=& -(0\,1)-(0\,2)-(0\,3)-(0\,4) \\
4x_1-x_2-x_3-x_4 &=& (0\,1)-(1\,2)-(1\,3)-(1\,4) \\
4x_2-x_3-x_4 &=& (0\,2)+(1\,2)-(2\,3)-(2\,4) \\
4x_3-x_4 &=& (0\,3)+(1\,3)+(2\,3)-(3\,4) \\
4x_4 &=& (0\,4)+(1\,4)+(2\,4)+(3\,4)
\end{array}\right\} \qquad (6)$$

Adding these equations there results

$$0 = 0.$$

The reason of the indeterminate form is that neither the interval nor the terminal point has been fixed. Consequently any arbitrary relation may be assumed between the corrections.

If as above we put $x_0 = 0$ and $x_4 = 0$ in the observation equations then the normal equations become

$$\left.\begin{array}{rcl}
4x_1-x_2-x_3 &=& (0\,1)-(1\,2)-(1\,3)-(1\,4) \\
4x_2-x_3 &=& (0\,2)+(1\,2)-(2\,3)-(2\,4) \\
4x_3 &=& (0\,3)+(1\,3)+(2\,3)-(3\,4)
\end{array}\right\}, \qquad (7)$$

from which x_1, x_2, x_3 may be found.

However, by assuming the arbitrary relation

$$x_0+x_1+x_2+x_3+x_4 = 0 \qquad (a)$$

between the corrections we obtain a much more simple solution. For by adding this relation to each of the normal equations (6) we have

$$\left.\begin{array}{rclcl}
5x_0 &=& -(0\,1)-(0\,2)-(0\,3)-(0\,4) &=& [al] \text{ suppose} \\
5x_1 &=& (0\,1)-(1\,2)-(1\,3)-(1\,4) &=& [bl] \quad \text{“} \\
5x_2 &=& (0\,2)+(1\,2)-(2\,3)-(2\,4) &=& [cl] \quad \text{“} \\
5x_3 &=& (0\,3)+(1\,3)+(2\,3)-(3\,4) &=& [dl] \quad \text{“} \\
5x_4 &=& (0\,4)+(1\,4)+(2\,4)+(3\,4) &=& [el] \quad \text{“}
\end{array}\right\}. \qquad (8)$$

The whole solution may therefore be conveniently arranged in tabular form. The sums of the horizontal rows are first formed and then placed in the proper vertical columns with the signs changed. The vertical columns are next added.

0	1	2	3	4	
—Sum$_1$	(0 1) —Sum$_2$	(0 2) (1 2) —Sum$_3$	(0 3) (1 3) (2 3) —Sum$_4$	(0 4) (1 4) (2 4) (3 4)	Sum$_1$ Sum$_2$ Sum$_3$ Sum$_4$
[*al*]	[*bl*]	[*cl*]	[*dl*]	[*el*]	

7. The preceding method of transforming the normal equations (6) into (8) by means of the arbitrary relation (a) is allowable. For since the sum of the coefficients of the unknowns in each of equations (6) is zero, whatever values of x_0, x_1, . . . satisfy these equations, the values $x_0+\lambda$, $x_1+\lambda$ . . . where λ is any constant will also satisfy them. Hence whatever set of values is taken to satisfy the equations the differences x_1-x_0, x_2-x_0, . . will be the same. Therefore by arbitrarily fixing the initial point of graduation determinate values are found for the corrections at the other points.

8. In the second place if, as is usually the case, the values of the spaces on the auxiliary bar are unknown the last of equations (4) becomes without meaning and must be omitted. This gives 9 observation equations involving 8 unknowns from which equations, as the initial graduation point is not fixed, the normal equations resulting would be indeterminate. Putting $x_0 = 0$ and $x_4 = 0$ we have 9 equations and 6 unknowns from which follow the normal equations

$$\left.\begin{array}{rrrrrrl}
+4x_1 & -\ x_2 & -\ x_3 & & +\ y_2 & +\ y_3 & = [al] \\
-\ x_1 & +4x_2 & -\ x_3 & & & & = [bl] \\
-\ x_1 & -\ x_2 & +4x_3 & & -y_2 & -\ y_3 & = [cl] \\
 & & & 4y_1 & & & = [dl] \\
+\ x_1 & & -\ x_3 & & +3y_2 & & = [el] \\
+\ x_1 & & -\ x_3 & & & +2y_3 & = [fl]
\end{array}\right\}, \qquad (9)$$

where $[al] = (0\,1)-(1\,2)-(1\,3)-(1\,4)$

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This is a more general solution of the problem.

The elimination of the normal equations from (4) may be considerably shortened by an artifice employed by Hansen in his treatise *Von der Bestimmung der Theilungsfehler eines gradlinigen Maassstabes,* but the amount of work is still very considerable.

9. The Precision of the values of the Unknowns, x_1, x_2, x_3.

Eliminating y_2, y_3 from the normal equations (9) the resulting equations may be written

$$\left.\begin{array}{r} 3\tfrac{1}{6}x_1 - x_2 - \tfrac{1}{6}x_3 = [al]' \\ 4x_2 - x_3 = [bl]' \\ +3\tfrac{1}{6}x_3 = [cl]' \end{array}\right\}. \qquad (10)$$

If the solution is finished by the Gaussian method of elimination the weights of the unknowns will be found to be

$$\text{weight of } x_1 = \tfrac{20}{7}, \text{ of } x_2 = \tfrac{10}{3}, \text{ of } x_3 = \tfrac{20}{7}.$$

Generally, if the number of graduation marks is $n+1$ the weights of the corrections to the two outside marks will be found to be $n(n+1) \div (2n-1)$ and of the center marks $\tfrac{2}{3}(n+1)$, where n is an odd number.

The p. e. of an observ'n of weight unity is found from the usual formula

$$r = .6745\sqrt{\frac{[vv]}{\text{no. of obs.—no. of indp't unknowns}}}$$

$$= .6745\sqrt{\frac{[vv]}{\frac{1}{2}(n-1)(n-2)}}, \qquad (11)$$

where v is the residual error of observation.

Hence the p. e. of x_1, x_2, x_3 are known, since their weights are known.

10. The solution of equations (9) is quite complicated. The amount of labor increases very rapidly with the increase of the number of spaces read on. Accordingly whenever it is not possible to make use of the simple and symmetrical form (8), an approximate form of solution should be employed. This is more especially true of thermometers where the carrying out of the rigorous solution would entail an expenditure of labor altogether out of proportion to the increased gain in accuracy. "To devote punctilious attention to hundredths of a thermometric degree in results derived from data which are uncertain by ten or twenty times that amount is to 'strain at a gnat and swallow a camel.'" (Amer. Jour. of Science, Feb., 1882.)

Instead of finding the corrections to the graduation marks directly, the corrections to the several spaces between the graduation marks may first be found and thence the cerrections to the marks at the ends of the spaces. Thus if z_1, z_2, ... denote the corrections to the spaces, then taking our example of four spaces

$$\left.\begin{array}{l} z_1 = x_1 - x_0 \\ z_2 = x_2 - x_1 \\ z_3 = x_3 - x_2 \\ z_4 = x_4 - x_3 \end{array}\right\}. \qquad (12)$$

Hence by addition

$$z_1 + z_2 + z_3 + z_4 = x_4 - x_0$$
$$= 0.$$

From the observation equations (4) we have by subtracting in pairs

$$\left.\begin{array}{l} z_2 - z_1 - [(1\,2)-(0\,1)] = v^{i} \\ z_3 - z_2 - [(2\,3)-(1\,2)] = v^{ii} \\ z_4 - z_3 - [(3\,4)-(2\,3)] = v^{iii} \\ z_3 - z_1 - [(1\,3)-(0\,2)] = v^{iv} \\ z_4 - z_2 - [(2\,4)-(1\,3)] = v^{v} \\ z_4 - z_1 - [(1\,4)-(0\,3)] = v^{vi} \end{array}\right\}.$$

Hence considering (1 2)—(0 1), (2 3)—(1 2), ... as the independently observed quantities (which is not correct as they are entangled), and making the sum of the squares of v^{i}, v^{ii}, . . . a minimum we have the normal eq'ns

$$\left.\begin{array}{l} 3z_1 - z_2 - z_3 - z_4 = -[(1\,2)-(0\,1)] - [(1\,3)-(0\,2)] - [(1\,4)-(0\,3)] \\ 3z_2 - z_3 - z_4 = +[(1\,2)-(0\,1)] - [(2\,3)-(1\,2)] - [(2\,4)-(1\,3)] \\ 3z_3 - z_4 = [(2\,3)-(1\,2)] - [(3\,4)-(2\,3)] - [(1\,3)-(0\,2)] \\ 3z_4 = [(3\,4)-(2\,3)] + [(2\,4)-(1\,3)] + [(1\,4)-(0\,3)] \end{array}\right\} \quad (14)$$

which when added give $0 = 0$ identically.

But $$z_1 + z_2 + z_3 + z_4 = 0.$$

Hence adding this relation to each of the normal equations we have the values of the unknowns. Thus

$$4z_1 = -[(1\,2)-(0\,1)] - [(1\,3)-(0\,2)] - [(1\,4)-(0\,3)]$$

$$- \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \quad - \qquad (15)$$

This form of approximation when applied to thermometers is known as Neumann's method. It is quite analogous to the system used by the Prussian Landestriangulation in the adjustment of horizontal angles and the principle involved is therefore really due to Gauss and Hansen.

For convenience the computation of the corrections may be arranged in tabular form as follows:

z_1	z_2	z_3	z_4	
$-\mathrm{Sum}_1$	$+(12)-(01)$ $-\mathrm{Sum}_2$	$+(13)-(02)$ $+(23)-(12)$ $-\mathrm{Sum}_3$	$+(14)-(03)$ $+(24)-(13)$ $+(34)-(23)$	Sum_1 Sum_1 Sum_3
$4z_1$	$4z_2$	$4z_3$	$4z_4$	

The corrections x_1, x_2, x_3 are then found from the relations

$$\left.\begin{array}{l} x_1 = z_1 \\ x_2 = x_1 + z_2 \\ x_3 = x_2 + z_3 \end{array}\right\} \quad (16)$$

From equations (15) and (16)

$$\left.\begin{array}{l} 4x_1 = -(12-01)-(13-02)-(14-03) \\ 4x_2 = -(13-02)-(14-03)-(23-12)-(24-13) \\ 4x_3 = -(14-03)-(24-13)-(34-23) \end{array}\right\} (17)$$

11. The Precision.—The p. e. of the unit of weight is given by

$$r = .6745\sqrt{\frac{[v'v']}{\frac{1}{2}(n-1)(n-2)}}, \qquad (18)$$

since the number of equations $= \frac{1}{2}n(n-1)$ and the number of independent unknowns $= n-1$, the number of spaces being n.

Also since z_1, z_2, z_3 result from the normal equations with the same w't they have the same p. e. From equations (17) it is evident that the p. e. of x_1 and x_3 is the same as that of z_1 but the p. e. of x_2 is $\sqrt{\frac{4}{3}}$ as great.

Generally when the number of spaces is an even number n, the p. e. of x_1 is to the p. e. of the center graduation mark as $\sqrt{(n-1)} : \frac{1}{2}n$.

Example.—The following were the observed values of the lengths of the 45°, 90° and 135° columns of Thermometer Green 4470, made to determine the calibration corrections at the 77°, 122° and 167° points. (Amer. Jour. Science, Vol. XXI, p. 374.)

45° col.	90° col.	135° col.
44.68	90.07	134.61
.71	.09	.68
.72	.11	
.75		

The corrections at the points 32° and 212° are assumed to be zero.

First Solution. By the method of Least Squares. The observation eq's are with the preceding notation, x_1, x_2, x_3 denoting the corrections at the 77°, 122° and 167° points respectively,

$$\begin{array}{rrrrrrrl}
 & & x_1 & -y_1 & & & -0.32 & = 0 \\
 & x_2 & -x_1 & -y_1 & & & -0.29 & = 0 \\
x_3 & -x_2 & & -y_1 & & & -0.28 & = 0 \\
-x_3 & & & -y_1 & & & -0.25 & = 0 \\
 & +x_2 & & & -y_2 & & +0.07 & = 0 \\
x_3 & & -x_1 & & -y_2 & & +0.09 & = 0 \\
 & -x_2 & & & -y_2 & & +0.11 & = 0 \\
x_3 & & & & & -y_3 & -0.39 & = 0 \\
 & & -x_1 & & & -y_3 & -0.32 & = 0
\end{array}$$

The Normal equations are

$$\begin{array}{rrrrrrl}
4x_1 & -\ x_2 & -\ x_3 & +\ y_2 & +\ y_3 & & = \ \ 0.33 \\
-\ x_1 & +4x_2 & -\ x_3 & & & & = \ \ 0.05 \\
-\ x_1 & -\ x_2 & +4x_3 & -\ y_2 & -\ y_3 & & = -0.20 \\
 & & & & & 4y_1 & = -1.14 \\
x_1 & & -\ x_3 & +3y_2 & & & = \ \ 0.27 \\
x_3 & & -\ x_3 & & +2y_3 & & = -0.71
\end{array}$$

Solving we find

$$\begin{aligned} x_1 &= +0.031 & y_1 &= —0.285 \\ x_2 &= +0.028 & y_2 &= 0.090 \\ x_3 &= +0.031 & y_3 &= —0.355 \end{aligned}$$

Substitute these values in the observation equations and the residuals result. Squaring and adding,

$$[vv] = 0.0003.$$

Hence for the p. e. of a single observation

$$r = .6745\sqrt{\frac{.0003}{3}} = \pm 0.0007.$$

$$\begin{aligned} \therefore \text{ p. e. of } x_1 &= .007\sqrt{.35} = 0^\circ.004 \\ \text{“ “ } x_2 &= .007\sqrt{.3} \ \ = 0^\circ.004 \\ \text{“ “ } x_3 &= .007\sqrt{.35} = 0^\circ.004 \end{aligned}$$

Second Solution. By Neumann's formula—Arranging according to table

z_1	z_2	z_3	z_4	Sums
	—.03	—.02	—.07	—.12
		—.01	—.02	—.03
			—.03	—.03
+.12	+.03	+.03		
+.12	00	00	—.12	

$$\begin{aligned} \therefore z_1 &= +0^\circ.03 & z_3 &= +0^\circ.00 \\ z_2 &= +0^\circ.00 & z_4 &= —0^\circ.03 \end{aligned}$$

$$\begin{aligned} \therefore x_1 &= z_1 &&= 0^\circ.03 \\ x_2 &= x_1+z_2 &&= 0\ .03 \\ x_3 &= x_2+z_3 &&= 0\ .03 \end{aligned}$$

Also

$$r = .6745\sqrt{\frac{.0004}{3}} = \pm 0.007$$

$$\begin{aligned} \text{and p. e. of } x_1 &= .007\sqrt{\tfrac{1}{3}} = \pm 0^\circ.004 \\ \text{“ “ } x_2 &= .007\sqrt{\tfrac{1}{4}} = \pm 0\ .004 \\ \text{“ “ } x_3 &= .007\sqrt{\tfrac{1}{3}} = \pm 0\ .004. \end{aligned}$$

THE MULTISECTION OF ANGLES.

BY PROF. J. W. NICHOLSON, A. M., LA. STATE UNIV., BATON ROUGE, LA.

To divide an angle into any number of equal parts, we use an instrument called the *multisector*, the simplest form of which is the square *sot;* which consists of two perpendicular arms *os* and *ct*, the length (*ct*) of the latter being equal to the width (*cp*)